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OPTIMAL SYSTEMS OF A COMBINATION OF CONTROL AND OBSERVATION

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We examine linear controllable systems under the condition of incomplete information on the current phase state, when at each instant only certain functions of the phase coordinates are accessible to measurement. We assume that noise, for which there is no special description, enters into the controllable system and into the measurement (observation) channel. Only the exact deterministic constraints which this noise satisfies, are known. At each instant the optimal controls are synthesized from the information on the whole previous history of observation on the basis of a minimax criterion for a convex function of phase coordinate values at the termination instant of the process. Thus, in the paper we examine a specific information game problem of conflict control [1]. The peculiarity of the problem manifests itself in the fact that here we are required to optimize as an aggregate both the synthesis process of the optimal control as well as the process of continuous estimation of the magnitudes of the current phase coordinates. The latter leads to the use of both the extremal constructions of the theory of differential-game problems of dynamics [1, 2] as well as the functional constructions for minimax problems of position observation and prediction [3, 4]. Fundamental attention is given to the constructive formation of the solution within the framework of convex analysis [5, 6]. The investigations in [7-9] were devoted to a different formulation of problems on the combination of control and observation.

1. Available information. Statement of the problem. Suppose that a controllable n-vector-valued quantity $x(t)$ varies in accordance with the equation

$$x^{\cdot}(t) = A(t)x(t) + B(t)u + C(t)v + f(t) \tag{1.1}$$

Here u and v are p- and q-dimensional controls, respectively, $f(t)$ is a known locally-integrable function. The m-dimensional quantity

$$y(t) = G(t)\, x(t) + F(t)\xi \tag{1.2}$$

is accessible to measurement at every instant t, beginning with the initial one equal to $t_0 - h$, $h > 0$, where ξ is the noise in the measurement (observation) channel. The system (1.1), (1.2) is examined on a fixed time interval $[t_0 - h, \vartheta]$. The system's coefficients are assumed continuous. The constraints

$$u[t] \in P, \qquad v[t] \in Q, \qquad \xi[t] \in R \tag{1.3}$$

for all t, where P, Q, R are convex compacta, known in advance in $R^{(p)}$, $R^{(q)}$, $R^{(r)}$, respectively, are imposed on the realizations $u[t]$, $v[t]$, $\xi[t]$ of the quantities u, v, ξ. It is essential that the realizations $v[t]$, $\xi[t]$ themselves are not given beforehand. Contrarily, the control u is subject to determination at each instant t; therefore, the realization $u_t[\cdot] = u[t+\sigma]$ can be taken as known. Here and later we adopt the notation $g_t[\cdot] = g[t+\sigma]$, $t_0 - h - t \leqslant \sigma \leqslant 0$, $g_t(\cdot) = g(t+\sigma)$.

We form the control u from the previous history of observation, i.e. from the function $y = y[t+\sigma]$ realized by virtue of (1.2), assuming that a "storage" of the quantities $y_t[\cdot]$, $u_t[\cdot]$ occurs. Farther on it is convenient to examine, instead of $u_t[\cdot]$, a function $z_t[\cdot]$, where $z[t]$ $(z(t))$ is a solution of the equation

$$z^{\cdot} = A(t)z + B(t)u[t] + f(t), \quad z(t_0 - h) = 0$$

uniquely determinable from $u[t]$ $(u(t))$. Thus, by the position of system (1.1), (1.2) we mean the quantity $\{t, \zeta_t(\cdot)\}$, where $\zeta_t(\cdot) = \{y_t[\cdot], z_t[\cdot]\}$. We seek control u as a functional of the position: $u = u(t, \cdot) = u(t, \zeta_t(\cdot))$. The class of admissible functionals $u(t, \cdot)$ is defined below.

Definition 1.1. The set of those and only those vectors $x \in R^{(n)}$ for each of which we can find functions $v(\tau)$, $\xi(\tau)$ satisfying (1.3), $t_0 - h \leqslant \tau \leqslant t$, such that the solution $y(\tau)$ of system (1.1), (1.2), found for $x = x(t)$, $v = v(\tau)$, $\xi = \xi(\tau)$, $u = u[\tau]$, on the interval $[t_0 - h, t]$ satisfies the condition $y(\tau) \equiv y[\tau]$, is called the region $X(t, \cdot) = X(t, \zeta_t(\cdot))$ admissible by position $\{t, \zeta_t(\cdot)\}$.

Suppose that we are given two functions $y_{t_1}^{(1)}(\cdot)$, $y_{t_2}^{(2)}(\cdot)$. We define the distance between them in the following way $(t_0 - h \leqslant \tau \leqslant t,\ t_1 \leqslant t_2)$:

$$d(y_{t_1}^{(1)}(\cdot), y_{t_2}^{(2)}(\cdot)) = \max_\tau \| y_*^{(1)}(\tau) - y^{(2)}(\tau) \|$$

$$y_*^{(1)}(\tau) \equiv y^{(1)}(\tau),\ \tau \in [t_0 - h, t_1]; \qquad y_*^{(1)}(\tau) \equiv y^{(1)}(t_1),\ \tau > t_1$$

Here $\|\cdot\|$ is the Euclidean norm. The distance $d(z_{t_1}^{(1)}(\cdot), z_{t_2}^{(2)}(\cdot))$ is defined analogously. The distance between the convex sets X_1, X_2 is defined with the aid of the Hausdorff metric.

Definition 1.2. The multivalued functional $U = U(t, X(t, \zeta_t(\cdot)))$ with values in the form of convex compacta contained in P, upper-semicontinuous by inclusion in $X(t, \cdot)$ for $\Delta t \geqslant 0$, is called an admissible x-strategy of the control.

The notion of the semicontinuity here is the standard one, taking into account the form of the metric in the set of sets $\{X\}$ We consider a subdivision of the interval $[t_0, \vartheta]$ into semi-intervals of the form $[\tau_i, \tau_{i+1})$, $\tau_{i+1} - \tau_i = \Delta_i > 0$, $t_0 = \tau_0$.

Definition 1.3. The single-valued functionals of the form $U(t, \tau_i, \zeta_{\tau_i}(\cdot))$ with values in P, continuous in t on the intervals of $t \in [\tau_i, \tau_{i+1})$ $(\Delta_i \leqslant \Delta$ for any $i)$, are called admissible Δ-strategies.

We denote by U_0, U_1, respectively, the strategy classes corresponding to Definitions 1.2, 1.3. Let $U(t, \cdot) = U(t, \zeta_t(\cdot))$ be one of the strategies mentioned. By a motion of system (1.1), (1.2), generated by $U(t, \cdot)$ and by a fixed pair of functions $v[t]$, $\xi[t]$, we mean all pairs of functions $x[t]$, $y[t]$ of the form

$$x[t] = p[t] + z[t], \qquad y[t] = q[t] + r[t] \tag{1.4}$$

where for almost all t

$$p^{\cdot}[t] = A(t)p[t] - C(t)v[t]; \quad q[t] = G(t)p[t] + F(t)\xi[t] \tag{1.5}$$

$$z^{\cdot}[t] = A(t)z(t] + B(t)u[t] + f(t) \tag{1.6}$$
$$r[t] = G(t)z[t], \quad z[t_0 - h] = 0, \quad u[t] \in U(t, \cdot)$$

For piecewise-program Δ-strategies the existence of a solution follows from the general properties of linear differential equations, and the following condition is valid for x strategies : a solution of system (1.4) -(1.6) exists if for $u[t] \equiv 0$ the multivalued function $X(t, \zeta_t(\cdot))$ is continuous in t, for any pair $v[t]$, $\xi[t]$, on the whole interval $[t_0 - h, \vartheta]$, except, perhaps, for only a countable set of points.

We define the initial position of the system. We assume that the formation of $u[t]$ can begin only at the instant t_0 (whereas the information on the function $y[t]$ comes in from the instant $t_0 - h$). The realization is taken as specified on $[t_0 - h, t_0]$. Here, for the sake of definiteness, we take $u[t] \equiv 0$, $t_0 - h \leqslant t \leqslant t_0$. One result of the assumption made is that the region $X(t_0, \zeta_{t_0}(\cdot))$ admissible by initial position $\zeta_{t_0}(\cdot) = \{y_{t_0}[\cdot], 0\}$ is already known at the start of the formation of control $u[t]$. This region is bounded if the homogeneous system (1.1), (1.2) is completely observable. (The assumptions indicated can be replaced by the condition $x(t_0) \in X_0$, where X_0 is a given bounded set).

Let $\Phi(X)$ be a proper convex functional given on the metric space $F = \{X\}$ of closed convex sets in $R^{(n)}$ with a Hausdorff metric. In particular, we can have (φ is a proper convex function in $R^{(n)}$ [5])

$$\Phi(X) = \max_x \varphi(x), \quad x \in X; \quad \min \varphi(x) = 0, \quad x \in R^{(n)} \tag{1.7}$$

We denote

$$g(\cdot) = g(t), \quad g[\cdot] = g[t], \quad t_0 - h \leqslant t \leqslant \vartheta \tag{1.8}$$
$$\Phi^\circ(t_0, \zeta_{t_0}(\cdot)) = \min_{U(t,\cdot)} \max_{x, v[\cdot], \xi[\cdot]} \max_{z[\cdot]} \Phi(X(\vartheta, \zeta_\vartheta(\cdot)))$$

Here $U(t, \cdot) \in U$, where U is one of the above-mentioned strategy classes U_0, U_1; $x \in X(t_0, \zeta_{t_0}(\cdot))$, $z[\cdot] \in Z(\cdot)$, where $Z(\cdot)$ is the set of trajectories of system (1.6) with $u[t] \in U(t, \cdot)$. Thus, each strategy $U(t, \cdot)$ generates a set of trajectories $z[\cdot]$ and each triple $\{x, v[\cdot], \xi[\cdot]\}$ generates the realization $y[\tau] - r[\tau]$, $t_0 \leqslant \tau \leqslant \vartheta$, namely, one of the possible continuations onto the interval $[t_0, \vartheta]$ of the signal $y[t_0 + \sigma] - r[t_0 + \sigma]$, $-h \leqslant \sigma \leqslant 0$. In its own turn, each pair $y_t[\cdot], z_t[\cdot]$ defines a point set $X(\vartheta, \zeta_\vartheta(\cdot))$. The paper's purpose is to ascertain existence conditions and to determine the optimal strategy $U^\circ(t, \cdot)$, ensuring the exact equality

$$\Phi^\circ(t_0, \zeta_{t_0}(\cdot)) = \max_{x, v[\cdot], \xi[\cdot]} \max_{z[\cdot]} \Phi(X(\vartheta, \zeta_\vartheta(\cdot))) \tag{1.9}$$
$$v[t] \in Q, \quad \xi[t] \in R \quad \text{for all } t$$
$$x \in X(t_0, \zeta_{t_0}(\cdot)), \quad z(\cdot) \in Z^\circ(\cdot) \quad (Z^\circ(\cdot) = Z(\cdot) \text{ for } U(t, \cdot) = U^\circ(t, \cdot))$$

or else an approximation of it.

2. Estimate of admissible regions. The solution of the problem presumes looking on the combined optimization both as a control process as well as an observation process, i.e. an estimate of the region of sojurn of the phase vector x at each current instant.

Suppose that on the interval $\tau \in [t_0 - h, t]$ we have realized a control $u_t[\cdot]$

generating by virtue of (1.6) a trajectory $z[\tau]$ and realized a function $y_t[\cdot]$ measured in the observation process. The realization $y_t[\cdot]$ could not arise for just any $v_t[\cdot]$, $\xi_t[\cdot]$, $x(t)$. Namely, knowledge of $y_t[\cdot]$ permits us to establish a posteriori the set of those and only those values of the phase vector $x = x(t)$ for each of which we can find measurable functions $v(\tau)$, $\xi(\tau)$, $\tau \in [t_0 - h, t]$ with values in Q and R, such that the elements $\{x(t) = x, v = v(\tau), \xi = \xi(\tau), u = u[\tau]\}$ together generate, by virtue of system (1.1), (1.2), precisely the realization $y(\tau) \equiv y[\tau]$. The stated values of $\{x\}$ can solely be compatible with the realization $y_t[\cdot]$. The set of vectors $\{x\}$ indicated forms the region $X(t, \zeta_t(\cdot)) = X(t, \cdot)$ admissible by position $\zeta_t(\cdot) = \{y_t[\cdot], z_t[\cdot]\}$ according to Definition 1.1.

The description of the region $X(t, \cdot)$, as well as the dynamics of its variation during the course of time, depending on the position $\{t, \zeta_t(\cdot)\}$ realized, comprise an important element in the solution. Here we stress that it is precisely the dependence of $X(t, \cdot)$ on $\{t, \zeta_t(\cdot)\}$ (and not on t alone) that leads to the problem of combined optimization of the control and observation processes. (In the contrary case we would have the problem of control when the current state is a convex set known beforehand).

By the symbol $\rho(l; Q) = \sup l'q$, $q \in Q$ (the prime denotes transposition), we denote the support function of the set Q [7]. A detailed derivation of the formula for $\rho(l; X(t, \cdot))$ and the description of the properties of set $X(t, \cdot)$ have been presented in [3, 4]. In them it was noted, in particular, that the sets $X(t, \cdot)$ are closed and convex, and conditions were derived ensuring the boundedness of $X(t, \cdot)$ for any position $\{t, \zeta_t(\cdot)\}$. The conditions mentioned reduce to the requirement of complete observability of system (1.1), (1.2) (with $v \equiv 0$, $\xi \equiv 0$, $u \equiv 0$) on any finite time interval, which requirement we take as fulfilled in what follows [4]. According to [3] we have

$$\rho(l; X(t, \cdot)) = l'z[t] + \psi(l, \zeta_t(\cdot), t, t_0 - h) \tag{2.1}$$

$$\psi(l, \zeta_t(\cdot), t, t_0 - h) = \inf\Big\{\int_{t_0-h}^{t} \rho(-s(\tau; \lambda(t, \cdot))C(\tau); Q)\,d\tau +$$

$$\int_{t_0-h}^{t} [\rho(\lambda(t, \tau)F(\tau); R) + \lambda(t, \tau)(y[\tau] - z[\tau])]\,d\tau$$

over all $\lambda(t, \cdot) \in \Lambda(t, l)$, where $\Lambda(t, l) = \{\lambda(t, \cdot) \equiv \lambda(t, \tau),\ t_0 - h \leqslant \tau \leqslant t\}$ is the set of all solutions of the equation

$$s(t; \lambda(t, \cdot)) = l \tag{2.2}$$

in the class of square-summable m-vector-valued functions: $\lambda(t, \tau) \in L_2^{(m)}$, $\tau \in [t_0 - h, t]$. Here $s(\tau; \lambda(t, \cdot))$ is the solution of the equation

$$ds / d\tau = -sA(\tau) + \lambda(t, \tau)G(\tau), \quad s(t_0 - h) = 0 \tag{2.3}$$

We consider the set

$$G(\vartheta, t, x, z[\vartheta \mid t]) = \bigcup_{v(\cdot)} \Big\{X(\vartheta, t)x + z[\vartheta \mid t] - \int_{t}^{\vartheta} X(\vartheta, \xi)C(\xi)v(\xi)\,d\xi\Big\} \tag{2.4}$$

$$\Big(z[t_1 \mid t] = \int_{t}^{t_1} X(t_1, \xi)(B(\xi)u[\xi] + f(\xi))\,d\xi\Big)$$

being the attainability region of system (1.1) from the state $x(t) = x$ in time $\vartheta - t$ under controls $v(\tau) \in Q$ for a fixed $u = u[\tau]$. Here $X(\vartheta, t)$ is the normed fundamental matrix of the homogeneous system (1.1). We denote

$$G(\vartheta, \tau, Q, z[\vartheta \mid t]) = \bigcup_{q \in Q} G(\vartheta, t, q, z[\vartheta \mid t]) \tag{2.5}$$

Further, let $y_{t_1}[\cdot \mid t]$ denote a segment of the realization of signal $y[\tau]$ of (1.2) on the interval $[t, t_1]$ (i.e. $y_{t_1}[\cdot \mid t] = y[t_1 + \sigma]$, $t - t_1 \leqslant \sigma \leqslant 0$, and let $z_{t_1}[\cdot \mid t]$ be the analogous segment for $z[\tau]$; let $X(t_1, \cdot \mid t) = X(t_1, \zeta_{t_1}(\cdot \mid t))$ be a region of the phase space, admissible by the quantities $\{y_{t_1}[\cdot \mid t], z_{t_1}[\cdot \mid t]\} = \zeta_{t_1}(\cdot \mid t)$ (i.e. satisfying Definition 1.1 wherein we should take $\zeta_{t_1}(\cdot \mid t))$ instead of $\zeta_{t_1}(\cdot)$).

Lemma 2.1. The equality

$$X(t_2, \zeta_{t_2}(\cdot \mid t_0)) = X(t_2, \zeta_{t_2}(\cdot \mid t_1)) \cap G(t_2, t_1, X(t_1, \zeta_{t_1}(\cdot \mid t_0)), z[t_2 \mid t_1]) \tag{2.6}$$

$(t_0 \leqslant t_1 \leqslant t_2)$ is valid.

We prove the validity of (2.6) by comparing the support functions of the convex compacta occurring in the left- and right-hand sides of this relation. According to (2.1) we have

$$\rho(l; X(t_2; \zeta_{t_2}(\cdot \mid t_0)) = l'z[t_2 \mid t_0] + \psi(l; \zeta_{t_2}(\cdot \mid t_0), t_2, t_0) \tag{2.7}$$

On the other hand, from (2.4), (2.5), [5] we obtain (in detail writing)

$$\begin{aligned} \rho(l; G(t_2, t_1, X(t_1, \zeta_{t_1}(\cdot \mid t_0)), z[t_2 \mid t_1])) = l'X(t_2, t_1) z[t_1] + \\ \inf_{\lambda(t_1, \cdot)} \Big\{ \int_{t_0}^{t_1} \rho(s(\xi; \lambda(t_1, \cdot))_{t_0} B(\xi); Q)\, d\xi + \\ \int_{t_0}^{t_1} [\lambda(t_1, \xi)(y[\xi] - z[\xi]) + \rho(\lambda(t_1, \xi) F(\tau); R)]\, d\xi \Big\} + \\ \int_{t_1}^{t_2} \rho(s(t_1; \lambda(t_1, \cdot))_{t_0} X(t_1, \xi) B(\xi); Q)\, d\xi \end{aligned} \tag{2.8}$$

over all $\lambda(t_1, \tau)$ satisfying the equality

$$s(t_1; \lambda(t_1, \cdot))_{t_0} = l'X(t_2, t_1) \tag{2.9}$$

Here $s(\tau; \lambda(t_1, \cdot))_{t_0}$ is the solution of (2.3) when $s(t_0) = 0$.

Analogously, from (2.1) we find

$$\begin{aligned} \rho(l; X(t_2, \zeta_{t_2}(\cdot \mid t_1))) = l'z[t_2 \mid t_1] + \\ \inf_{\lambda(t_2, \cdot)} \Big\{ \int_{t_1}^{t_2} [\rho(s(\xi; \lambda(t_2, \cdot))_{t_1} B(\xi); Q) + \\ \lambda(t_2, \xi)[y[\xi] - z[\xi]) + \rho(\lambda(t_2, \xi) F(\xi); R)]\, d\xi \Big\} \end{aligned} \tag{2.10}$$

over all $\lambda(t_2, \tau)$ satisfying the equality

$$s(t_2; \lambda(t_2, \cdot))_{t_1} = l' \tag{2.11}$$

Now, taking the structure of the right-hand side of (2.6), equalities (2.7) – (2.11), and the formula [7]

$$\rho\,(l;\,Q_1 \cap Q_2) = \inf_{l_1 l_2}\ \{\rho\,(l_1;\,Q_1) + \rho\,(l_2;\,Q_2)\},\quad l_1 + l_2 = l$$

into consideration, we arrive at once at the support function of the right-hand side of (2.6), coinciding with expression (2.8).

Corollary 2.1. The condition

$$G\,(\vartheta,\,t_1,\,X\,(t_1,\,\cdot),\,z\,[\vartheta \mid t_1]) \supseteq G\,(\vartheta,\,t_2,\,X\,(t_2,\,\cdot),\,z\,[\vartheta \mid t_2]) \tag{2.12}$$

$(t_2 \geqslant t_1)$ is valid for arbitrary realization $u\,[t]$, $t_0 \leqslant t \leqslant \vartheta$, generating the quantities $z\,[\vartheta \mid t_1]$, $z\,[\vartheta \mid t_2]$.

In fact, on the one hand we have the valid equality [1]

$$G\,(\vartheta,\,t_1,\,X\,(t_1,\,\cdot),\,z\,[\vartheta \mid t_1]) = \\ G\,(\vartheta,\,t_2,\,G\,(t_2,\,t_1,\,X\,(t_1,\,\cdot),\,z\,[t_2 \mid t_1]),\ z\,[\vartheta \mid t_2])$$

and on the other, according to Lemma 2.1,

$$X\,(t_2,\,\cdot) \subseteq G\,(t_2,\,t_1,\,X\,(t_1,\,\cdot),\,z\,[t_2 \mid t_1])$$

Condition (2.12) now follows from the obvious inclusion (see (2.5))

$$G\,(\vartheta,\,t_2,\,Q_1,\,z\,[t_2 \mid t_1]) \supseteq G\,(\vartheta,\,t_2,\,Q_2,\,z\,[t_2 \mid t_1]),\quad Q_1 \supseteq Q_2$$

We say that a function $G\,(t)$ (G are convex sets) has a jump at $t = t_1$ if the function $\rho\,(l;\,G\,(t))$ has a jump at the point $t = t_1$ even if for one l. From the convexity and compactness of sets $G\,(\vartheta,\,t,\,X\,(t,\,\cdot),\,z\,[\vartheta \mid t])$ we arrive at the following conclusion (see Lemma 2.1).

Lemma 2.2. The function $G\,(\vartheta,\,t,\,X\,(t,\,\cdot),\,z\,[\vartheta \mid t]) = G\,[t]$ has no more than a countable set of jumps.

3. An auxiliary prediction problem. Suppose that at an instant $t \in [t_0, \vartheta]$ we have realized the position $\{t,\,\zeta_t\,(\cdot)\}$, $\zeta_t\,(\cdot) = \{y_t\,[\cdot],\,z_t\,[\cdot]\}$, generating the set $X\,(t,\,\cdot)$ (admissible by this position). Let us fix a triple of functions $u^*\,(\tau)$, $v^*\,(\tau)$, $\xi^*\,(\tau)$, $t \leqslant \tau \leqslant \vartheta$ with values in P, Q, R, respectively. Then $u^*(\tau)$ generates the realization $z_\vartheta{}^*\,(\,\cdot \mid t)$, namely, a segment of the solution on $[t, \vartheta]$ of Eq. (1.6) with $u\,[\tau] = u^*\,(\tau)$. In turn, $z_\vartheta{}^*\,(\cdot \mid t)$, $v^*\,(\tau)$, $\xi^*\,(\tau)$ generate the set of realizations $Y\,(\cdot \mid t) = \{[y_\vartheta{}^*\,(\cdot \mid t) \mid x]\}$, $x \in X\,(t,\,\cdot)$. Here $[y_\vartheta{}^*\,(\cdot \mid t) \mid x]$ is the solution on $[t, \vartheta]$ of system (1.1), (1.2) with $u = u^*\,(\tau)$, $v = v^*\,(\tau)$, $\xi = \xi^*\,(\tau)$, $x\,(t) = x$. Thus, each of the functions $[y_\vartheta{}^*\,(\cdot \mid t) \mid x]$ is one of the possible continuations of realization $y_t\,[\cdot]$ onto the interval $(t, \vartheta]$ when the quantity $z^*_\vartheta\,(\cdot \mid t)$ is simultaneously realized on it. Here each of the mentioned possible continuations of signal $y_t\,[\cdot]$ (i.e. each of the elements of $Y\,(\cdot \mid t)$) is generated by the functions $v^*\,(\tau)$, $\xi^*\,(\tau)$, and by one of the vectors $x \in X\,(t,\,\cdot)$.

We introduce the notation

$$[\zeta_\vartheta{}^*\,(\cdot \mid t) \mid x] = \{[y_\vartheta{}^*\,(\cdot \mid t) \mid x],\ z_\vartheta{}^*\,(\cdot \mid t)\}$$

$$[\zeta_\vartheta{}^*\,(\cdot) \mid x] = \{y_\vartheta{}^*\,(\cdot),\ z_\vartheta{}^*\,(\cdot)\}$$

$$y^*\,(\vartheta + \sigma) \equiv y\,[\vartheta + \sigma],\qquad t_0 - h - \vartheta \leqslant \sigma < t - \vartheta$$

$$y^*\,(\vartheta + \sigma) \equiv [y^*\,(\vartheta + \sigma \mid t) \mid x],\qquad t - \vartheta \leqslant \sigma \leqslant 0$$

$$z^*\,(\vartheta + \sigma) \equiv z\,[\vartheta + \sigma],\qquad t_0 - h - \vartheta \leqslant \sigma < t - \vartheta$$

$$z^*\,(\vartheta + \sigma) \equiv z^*\,(\vartheta + \sigma \mid t),\qquad t - \vartheta \leqslant \sigma \leqslant 0$$

If at instant t we have realized the set $X(t, \cdot)$, then (for $u = u^*(\tau)$, $v = v^*(\tau)$, $\xi = \xi^*(\tau)$, $\tau \in [t, \vartheta]$) at the instant ϑ we can make an a priori realization of any of the sets $X(\vartheta, \cdot \mid x) = X(\vartheta, [\zeta_\vartheta^*(\cdot) \mid x])$. According to Lemma 2.1 we have

$$X(\vartheta, \cdot \mid x) = X(\vartheta, [\zeta_\vartheta^*(\cdot \mid t) \mid x]) \cap G(\vartheta, t, X(t, \cdot), z^*(\vartheta \mid t)) \quad (3.1)$$

Let us find the support function of set $X(\vartheta, \cdot \mid x)$. Using the concepts presented in Sect. 2, we find

$$\rho(l, X(\vartheta, \cdot \mid x)) = l'z^*(\vartheta \mid t) + l'X(\vartheta, t)x + \quad (3.2)$$
$$\rho(l; X(\vartheta, t)G^{(1)}(t, x) \cap G^{(2)}(\vartheta, t, v^*(\cdot) \ \xi^*(\cdot)))$$

$$G^{(1)}(t, x) = X(t, \cdot) - x$$

$$\rho(l; G^{(2)}(\vartheta, t, v^*(\cdot), \xi^*(\cdot)) =$$
$$\inf_{\lambda(\vartheta,\cdot)} \ \sup_{v(\cdot),\xi(\cdot)} \int_t^\vartheta [s(\tau; \lambda(\vartheta, \cdot))_t C(\tau)(v^*(\tau) - v(\tau)) +$$
$$\lambda(\vartheta, \tau)F(\tau)(\xi^*(\tau) - \xi(\tau))]d\tau \quad (s(\vartheta; \lambda(\vartheta, \cdot))_t = l')$$

Formula (3.2) yields an exact solution of the prediction problem, namely, a description of all a priori possible realizations $X(\vartheta, \cdot \mid x)$ of set $X(\vartheta, \cdot)$, if $X(t, \cdot)$ is realized at instant t.

Let us consider the problem of the program maximin of functional Φ

$$\varepsilon^\circ(t, \cdot) = \varepsilon^\circ(t, \zeta_t(\cdot)) = \max_{x,v^*,\xi^*} \quad \min_{u^*}\Phi(X(\vartheta, \cdot \mid x)) \quad (3.3)$$
$$x \in X(t, \cdot), \quad v^*(\tau) \in Q, \quad \xi^*(\tau) \in R, \quad u^*(\tau) \in P, \quad t \leqslant \tau \leqslant \vartheta$$

The quantity $\varepsilon^\circ(t, \cdot)$ is the best guaranteed result from functional Φ if at instant t we pass to a purely program control, having received no additional information whatever on position $\{\tau, \zeta_\tau(\cdot)\}$ for $\tau > t$. We define the functional $\Phi(X)$ concretely later on by choosing it in form (1.7). Solving problem (3.3) with the aid of formula (3.2), we arrive at the assertion

$$\varepsilon^\circ(t, \cdot) = \sup_l \{\Psi(t, X(t, \cdot), l) - \varphi^*(-l)\}, \quad l \in R^{(n)} \quad (3.4)$$

$$\Psi(t, X(t, \cdot), l) = \int_t^\vartheta (\rho(l'X(\vartheta, \tau)C(\tau); Q) -$$
$$\rho(l'X(\vartheta, \tau)B(\tau); P))d\tau + \rho(-l'X(\vartheta, t)X(t, \cdot))$$

Here $\varphi^*(l)$ is the function adjoint to $\varphi(l)$, so that $\varphi(x) = \sup(l'x - \varphi^*(l))$ over all $l \in R^{(n)}$ [5]. We note that the equality

$$\Psi(t, X(t, \cdot), -l) = \min_{u^*} \rho(l; G(\vartheta, t, X(t, \cdot), z^*(\vartheta \mid t)) \quad (3.5)$$

$(u^*(\tau) \in P)$ is valid.

Note 3.1. Let $\Phi(X) = r_\varphi(X)$, where $r_\varphi(X) = \min_y \max_x \varphi(x - y)$, $x, y \in X$ is the so-called Chebyshev φ-radius of set X (φ is a proper convex function). In particular, if $\varphi(x) = (x'x)^{1/2}$, then $r_\varphi(X) = d(X)$, where $d(X) = \max_l (\rho(l; X) - \rho(-l; X))$ is the diameter of X [5]. It follows from formulas (2.1), (3.2) (also see [3, 4]) that the quantities $r_\varphi(X(\vartheta, \cdot))$, $r_\varphi(X(\vartheta; [\zeta^*(\cdot \mid t) \mid x])$ do not depend upon the choice of control u. We note that a formula for $r_\varphi(X(\vartheta\ \cdot))$ was derived in [4] when $\xi \equiv 0$

4. Fundamental estimate. Suppose that in time Δt system (1.1), (1.2) is carried from position $\{t,\ \zeta_t(\cdot)\}$ to the position $\{t+\Delta t,\ \zeta_{t+\Delta t}(\cdot)\}$. To solve the problem we should estimate the increment

$$\Delta\varepsilon^\circ(t,\ \cdot) = \varepsilon^\circ(t+\Delta t,\ \zeta_{t+\Delta t}(\cdot)) - \varepsilon^\circ(t,\ \zeta_t(\cdot)) \tag{4.1}$$

As a preliminary we prove a number of auxiliary assertions.

Consider the function $\gamma(t, X, l) = \Psi(t, X, l) - \varphi^*(-l)$. We denote

$$\Gamma(t,\ \cdot) = \Gamma(t,\ \zeta_t(\cdot)) = \{l^\circ : \sup_l \gamma(t, X(t,\ \cdot), l) = \gamma(t, X(t,\ \cdot), l^\circ)\}$$

The boundedness of $\Gamma(t,\ \cdot)$ is ensured by the following additional assumption which we adopt in what follows:

Condition 4.1. For the arbitrary number $N > 0$ and the arbitrary convex everywhere positive-definite homogeneous function $\beta(l)$, there exists a number $N_1(N;\ \beta(l))$ such that the inequality $\|l\| \leqslant N_1$ holds for $-\gamma \leqslant N,\ \rho(l;\ X) \leqslant \beta(l)$.

Condition 4.1 is a constraint on the function $\varphi(l)$. Let $\varphi(x)$ be the Euclidean distance $r(x, M)$ from x to a convex set M. Then

$$\varphi(x) = \max_l (l'x - r(l; M)), \qquad \|l\| = 1$$

moreover,

$$\varphi^*(l) = r(l; M), \quad \|l\| \leqslant 1, \quad \varphi^*(l) = \infty, \quad \|l\| > 1 \tag{4.2}$$

Hence it follows that Condition 4.1 is satisfied when $\varphi(x) = r(x, M)$. We say that the functional $F(t,\ \zeta_t(\cdot)) = F(t,\ \cdot)$ is continuous along (direction)

$$\{\Delta t,\ \Delta\zeta_t(\cdot)\} = \{\Delta t,\ \zeta_{t+\Delta t}(\cdot) - \zeta_t(\cdot)\}$$

at point $\{t,\ \zeta_t(\cdot)\}$, if

$$F(t+\lambda\Delta t,\ \zeta_{t+\lambda\Delta t}(\cdot)) \to F(t,\ \zeta_t(\cdot)),\ \lambda \to 0$$

Lemma 4.1. Let functional $X(t,\ \cdot)$ be continuous along $\{\Delta t,\ \Delta\zeta_t(\cdot)\}$ at point $\{t,\ \zeta_t(\cdot)\}$. Then at this point the functional $\varepsilon^\circ(\tau,\ \cdot)$ is continuous along $\{\Delta t,\ \Delta\zeta_t(\cdot)\}$.

From (3.4) it follows that the functional $\gamma(t, X(t,\ \cdot), l)$ is continuous along $\{\Delta t,\ \Delta\zeta_t(\cdot)\}$ at point $\{t,\ \zeta_t(\cdot)\}$. We have $(0 < \lambda < 1)$

$$\gamma(t+\lambda\Delta t, X(t+\lambda\Delta t,\ \cdot), l_{\lambda\Delta}) = \varepsilon^\circ(t+\lambda\Delta t,\ \cdot) \geqslant \gamma(t+\lambda\Delta t, X(t+\lambda\Delta t,\ \cdot), l^\circ) \tag{4.3}$$

$$\gamma(t, X(t,\ \cdot), l_{\lambda\Delta}) \leqslant \varepsilon^\circ(t,\ \cdot) = \gamma(t, X(t,\ \cdot), l^\circ) \tag{4.4}$$
$$l^\circ \in \Gamma(t,\ \cdot), \qquad l_{\lambda\Delta} \in \Gamma(t+\lambda\Delta t)$$

By virtue of the continuity of $X(\tau,\ \cdot)$ along $\{\Delta t,\ \Delta\zeta_t(\cdot)\}$, for $\sigma > 0$ we have

$$\gamma(t+\lambda\Delta t, X(t+\lambda\Delta t,\ \cdot), l^\circ) \geqslant \varepsilon^\circ(t,\ \cdot) - \sigma$$

provided $\lambda \leqslant \delta(\sigma)$.

Let $\beta_\Delta(l) = \max\ \rho(l;\ X(t+\Delta t,\ \cdot))$ over all sets $X(t+\Delta t,\ \cdot)$ into which $X(t,\ \cdot)$ can be carried in time Δt under the constraints (1.3) on u, v, ξ. Then, by Condition 4.1 $(N = \varepsilon^\circ(t,\ \cdot) - \delta)$ we obtain $\|l_{\lambda\Delta}\| \leqslant K$, provided

$$\lambda \leqslant \delta(\sigma), \qquad \rho(l;\ X(t+\lambda\Delta t;\ \cdot)) \leqslant \beta_\Delta(l)$$

The continuity of $\varepsilon^\circ(t, \cdot)$ along $\{\Delta t, \Delta\zeta_t(\cdot)\}$ at point $\{t, \zeta_t(\cdot)\}$ follows from the compactness of set $\| l_{\lambda\Delta} \| \leqslant K$, from the continuity of $X(t, \cdot)$ along $\{\Delta t, \Delta\zeta_t(\cdot)\}$, and from inequalities (4.3), (4.4).

Lemma 4.2. Let functional $\varepsilon^\circ(\tau, \cdot)$ be continuous along $\{\Delta t, \Delta\zeta_t(\cdot)\}$ at point $\{t, \zeta_t(\cdot)\}$. Then the set $\Gamma(\tau, \cdot)$ is upper-semicontinuous along $\{\Delta t, \Delta\zeta_t(\cdot)\}$ at this point (i.e. for any $\sigma > 0$ there exists $\delta > 0$ such that the inclusion $\Gamma(t + \lambda\Delta t, \cdot) \subset \Gamma_\sigma(t, \cdot)$ holds for $\lambda \leqslant \delta$, where Γ_σ is the σ-neighborhood of Γ).

Lemma 4.3. Suppose that at each point $\{\tau, \zeta_\tau(\cdot)\}$ of the region $\varepsilon^\circ(\tau, \cdot) > 0$ the set $\Gamma(\tau, \cdot)$ consists of a single element $l^\circ(\tau, \zeta_\tau(\cdot)) = l^\circ(\tau, \cdot)$. Then the functional $l^\circ(\tau, \cdot)$ is continuous along $\{\Delta t, \Delta\zeta_t(\cdot)\}$ at point $\{t, \zeta_t(\cdot)\}$, if $\varepsilon^\circ(\tau, \cdot)$ is continuous along direction $\{\Delta t, \Delta\zeta_t(\cdot)\}$ at this point.

The proof of Lemmas 4.3, 4.3, using Condition 4.1, follows the standard scheme in [1]. The next assertion follows from the properties of convex functions.

Lemma 4.4. For the hypotheses of Lemma 4.3 to be fulfilled it suffices to satisfy one of the following conditions:

A. The function $-\gamma(\tau, X(\tau, \cdot), l)$ is strictly convex in l.

B. $\varphi(x) = r(x, M)$, where M is a convex set, and the function $-\gamma(\tau, X(\tau, \cdot), l)$ is convex in l.

Let us pass on to the estimate of increment (4.1). In the notation of Sect. 3 let

$$\zeta_{t+}{}^*{}_{\Delta t}(\cdot) = [\zeta^*{}_{t+\Delta t}(\cdot) \mid x^*], \qquad \zeta_t(\cdot) = \{y_t[\cdot], z_t[\cdot]\}$$

By formula (3.4) we find

$$\Delta\varepsilon^\circ(t, \lambda, \cdot) = \varepsilon^\circ(t + \lambda\Delta t, \cdot) - \varepsilon^\circ(t, \cdot) = \gamma(t + \lambda\Delta t, X(t + \lambda\Delta t, \cdot), l_{\lambda\Delta}) - \gamma(t, X(t, \cdot), l^\circ) \tag{4.5}$$

whence for any l°, $l_{\lambda\Delta}$, $0 \leqslant \lambda \leqslant 1$

$$\Delta\varepsilon^\circ(t, \lambda, \cdot) \geqslant \Delta\gamma(t, \lambda, l^\circ,)\ \Delta\varepsilon^\circ(t, \lambda, \cdot) \leqslant \Delta\gamma(t, \lambda, l_{\lambda\Delta}) \tag{4.6}$$

$$\Delta\gamma(t, \lambda, l) = \gamma(t + \lambda\Delta t, X(t + \lambda\Delta t, \cdot), l) - \gamma(t, X(t, \cdot), l)$$

The increment $\Delta\varepsilon^\circ(t, 1, \cdot)$ corresponds to the realization $y^*(\tau)$, $z^*(\tau)$, generated on the interval $\tau \in [t, t + \Delta t]$ by the quantities $u^*(\tau)$, $v^*(\tau)$, $\xi^*(\tau)$. In order to write out explicit expressions for the right-hand sides of inequalities (4.6), we estimate the increment

$$\Delta\rho(t, \lambda, l, \cdot) = \rho(-l;\ \mathrm{X}(\vartheta, t + \lambda\Delta t) X(t + \lambda\Delta t, \cdot)) - \rho(-l;\ \mathrm{X}(\vartheta, t + \lambda\Delta t) X(t, \cdot))$$

for which we make use of formulas (3.1), (3.2) with $\vartheta = t + \lambda\Delta t$. By direct calculations we obtain

$$\Delta\rho(t, \lambda, l, \cdot) = -l'\mathrm{X}(\vartheta, t + \lambda\Delta t)\Big[z^*(t + \lambda\Delta t \mid t) - \int_t^{t+\lambda\Delta t} \mathrm{X}(t + \lambda\Delta t, \xi) C(\xi) v^*(\xi)\, d\xi\Big] - l'\mathrm{X}(\vartheta, t) x^* - \rho(-l'\mathrm{X}(\vartheta, t);\ X(t, \cdot)) + \omega(l, v_\lambda{}^*(\cdot), \xi_\lambda{}^*(\cdot), X^*, t, \lambda) \tag{4.7}$$

$$f_\lambda(\cdot) = f(\tau), \qquad t \leqslant \tau \leqslant t + \lambda\Delta t$$

$$\omega\,(l,\ v^*_\lambda\ (\cdot),\ \xi^*_\lambda\ (\cdot),\ x^*,\ t,\ \lambda) =$$
$$\rho\,(-l;\ G^{(1)}\,(t,\ x^*)\ \bigcap\ G^{(2)}\,(t+\lambda\Delta t,\ t,\ v^*\ (\cdot),\ \xi^*\ (\cdot)))$$

Let

$$G^*\ (t,\ \lambda,\ x^*,\ v^*_\lambda,\ (\cdot),\ \xi^*_\lambda\ (\cdot)) = \qquad (4.8)$$
$$G^{(1)}\,(t,\ x^*)\ \bigcap\ G^{(2)}\,(t+\lambda\Delta t,\ t,\ v^*\ (\cdot),\ \xi^*\ (\cdot)) + x^*$$

The condition

$$x^* \subseteq G^*\ (t,\ \lambda,\ x^*,\ v^*_\lambda\ (\cdot),\ \xi^*_\lambda\ (\cdot)) \subseteq X\ (t,\ \cdot) \qquad (4.9)$$

is valid by construction. From formulas (3.4), (4.7) – (4.9) we find

$$\Delta\gamma\,(t,\lambda,l) = \int\limits_t^{t+\lambda\Delta t} f\,(\tau,\ u^*(\tau),\ v^*(\tau),\ l)\,d\tau + \alpha^*\,(t,\lambda,l)$$

$$f\ (\tau,\ u,\ v,\ l) = \rho\ (l',X\ (\vartheta,\ \tau)B\ (\tau);\ \ P) - \rho\ (l'X\ (\vartheta,\ \tau)C\ (\tau);\ Q) - \qquad (4.10)$$
$$- l'X\,(\vartheta,\ \tau)\ (B\ (\tau)u^*\ (\tau) - C\ (\tau)v^*\ (\tau))$$

$$\alpha^*\ (t,\ \lambda,\ l) = \rho\ (-l;\ G^*\ (t,\ \lambda,\ x^*,\ v^*_\lambda\ (\cdot),\ \xi^*_\lambda\ (\cdot))) -$$
$$- \rho\ (-l;\ X\ (t,\ \cdot))$$

Let $\varepsilon^\circ\ (t,\ \cdot)$ be continuous along $\{\Delta t,\ \Delta\zeta_t\ (\cdot)\}$ at point $\{t,\ \zeta_t\ (\cdot)\}$. Then by Lemma 4.2 the set $\Gamma\ (t,\ \cdot)$ is upper-semicontinuous along $\{\Delta t,\ \Delta\zeta_t\ (\cdot)\}$ at this point. Hence we arrive at the following estimate.

Lemma 4.5. Let $\varepsilon^\circ\ (t,\ \cdot) > 0$. Then for any $\sigma > 0$ there exists $\delta > 0$ such that

$$f\ (\tau,\ u,\ v,\ l_{\lambda\Delta}) \leqslant \max_{l^\circ}\ (f\ (\tau,\ u,\ v,\ l^\circ) + \sigma,\ l^\circ \in \Gamma\ (t,\ \cdot) \qquad (4.11)$$

for arbitrary $l_{\lambda\Delta} \in \Gamma\ (t+\lambda\Delta t,\ \cdot),\ \tau \in [t,\ t+\lambda\Delta t],\ \lambda \leqslant \delta$. Estimate (4.11) is uniform in all $u \in P,\ v \in Q,\ \xi \in R$ and in all continuations $\zeta_{t+\Delta t}\ (\cdot)$ of realization $\zeta_t\ (\cdot)$ under constraints (1.3), for which $\varepsilon^\circ\ (t,\ \cdot)$ is continuous along $\{\Delta t,\ \Delta\zeta_t\ (\cdot)\}$.

We note that the condition

$$\max_x \rho\ (-l;\ G^*\ (t,\ \lambda,\ x,\ v^*_\lambda\ (\cdot),\ \xi^*_\lambda\ (\cdot))) = \qquad (4.12)$$
$$\rho\ (-l;\ X\ (t,\ \cdot)),\ x \in X\ (t,\ \cdot)$$

following from (4.8), (4.9) is valid. Inequalities (4.6), formula (4.10), and Lemma 4.5 lead to the estimates $(\lambda \to 0,\ l^\circ \in \Gamma\ (t,\ \cdot))$

$$\liminf \lambda^{-1}\Delta\varepsilon^\circ\ (t,\ \lambda,\ \cdot) \geqslant f\ (t,\ u^*\ (t),\ v^*\ (t),\ l^\circ) + \qquad (4.13)$$
$$\limsup \lambda^{-1}\alpha\ (t,\ \lambda,\ l^\circ)$$
$$\limsup \lambda^{-1}\Delta\varepsilon^\circ\ (t,\ \lambda,\ \cdot) \leqslant \max_{l^\circ} f\ (t,\ u^*\ (t),\ v^*\ (t),\ l^\circ) +$$
$$\liminf \lambda^{-1}\ (\max_{l^\circ} \alpha\ (t,\ \lambda,\ l^\circ))$$

If $\varepsilon^\circ\ (t,\ \cdot)$ is continuous along $\{\Delta t,\ \Delta\zeta_t\ (\cdot)\}$, then the condition

$$\max_{l^\circ} \{-l^{\circ\prime}x^* - \rho\ (-l^\circ;\ X\ (t,\ \cdot))\} = 0, \qquad l^\circ \in \Gamma\ (t,\ \cdot)$$

is satisfied, which together with (4.12) and (4.13) lead to the equality

$$\left.\frac{d\varepsilon^\circ\ (t,\ \cdot)}{dt}\right|_{u^*,\ v^*} = \max_{l^\circ} f\,(t,\ u^*\,(t),\ v^*\,(t),\ l^\circ), \qquad l^\circ \in \Gamma\,(t,\ \cdot) \qquad (4.14)$$

(after passing to the limit as $\lambda \to 0$) in (4.13)). Here the derivative of $\varepsilon^\circ\ (t,\ \cdot)$ along the realization $\{t,\ \zeta_t\ (\cdot)\}$ generated by controls $u^*\ (t),\ \ v^*\ (t)$ stands on the left-hand side. The equality cited is an analog of a well-known formula for the directional derivative of a function of the maximum [10, 11], depending in the case being considered on a specific argument $\zeta^*(\cdot)$. Its application to a linear differential game with complete information was discussed in [12, 13]. Formula (4.14) is transformed in an obvious manner if $\Gamma(t, \cdot)$ consists of a single element.

If $\{t, \zeta_t(\cdot)\}$ is a point of discontinuity of $\varepsilon^\circ(t, \cdot)$ along $\{\Delta t, \Delta\zeta_t(\cdot)\}$, then the inequality $\Delta\rho(t, \lambda, l, \cdot) \leqslant -\sigma \| l \|$, $\sigma > 0$ is valid for arbitrary $\lambda \leqslant \delta(\sigma)$. Then, the estimate

$$\lim_{\lambda\to 0} \quad \varepsilon^\circ(t + \lambda\Delta t, \cdot) < \varepsilon^\circ(t, \cdot) - \sigma_1, \sigma_1 > 0 \tag{4.15}$$

is valid independently of the choice of u, v, ξ at point t, and the functional has a jump of finite magnitude at point $\{t, \zeta_t(\cdot)\}$.

Condition 4.2. The condition

$$\min_u \max_v \left.\frac{d\varepsilon^\circ(t, \cdot)}{dt}\right|_{u, v} \leqslant 0 \tag{4.16}$$

is satisfied in the region $\varepsilon^\circ(t, \cdot) > 0$.

Condition 4.2 is automatically satisfied if $\Gamma(t, \cdot)$ consists of a single vector for each position $\{t, \zeta_t(\cdot)\}$ (Lemma 4.4). For a suitable choice of $u^\circ = u(t)$ this condition ensures the estimate

$$\varepsilon^\circ(t + \Delta t, \cdot) \leqslant \varepsilon^\circ(t, \cdot) + o(\Delta t) \tag{4.17}$$

uniform in all $t \in [t_0, \vartheta]$ and in all positions $\zeta_t(\cdot)$ from a closed bounded subset of the region $\varepsilon^\circ(t, \cdot) > 0$ $(o(\Delta t)(\Delta t)^{-1} \to 0$ as $\Delta t \to 0)$. In fact, suppose that the quantities $v = v^*(\tau)$, $\xi = \xi^*(\tau)$ are realized on the interval $\tau \in [t, t + \Delta t]$. Let

$$g_* = (\Delta t)^{-1} \int_0^{\Delta t} g(t + \tau)\, d\tau$$

From the vectors $v_*^* \in Q$, $\xi_*^* \in R$ we choose the vector $u_* \in P$ ensuring the inequality $d\varepsilon^\circ(t, \cdot)/dt \leqslant 0$. We arrive at the required inequality (4.17) by now setting $u^*(\tau) \equiv u_*$ and using (4.9), (4.14), the second of inequalities (4.6), Lemma 4.5, and condition (4.12). Summing conditions (4.15), (4.16) we arrive at the next assertion.

Lemma 4.6. Suppose that Conditions 4.1, 4.2, and the condition $\varepsilon^\circ(t, \cdot) > 0$ are satisfied and that the quantities x^*, $v^*(\tau)$, $\xi^*(\tau)$ have been realized on the interval $\tau \in [t, t + \Delta t]$. Then a control $u^*(\tau) \equiv u_*$ exists such that estimate (4.17), uniform in all $t \in [t_0, \vartheta]$ and in all positions $\zeta_t(\cdot)$ from a closed bounded subset of the region $\varepsilon^\circ(t, \cdot) > 0$ is valid for the new position $\{t + \Delta t, \zeta_{t+\Delta t}(\cdot)\}$ generated by the quantities x^*, $u^*(\tau)$, $v^*(\tau)$, $\xi^*(\tau)$.

. A condition analogous to the property of stability of program absorption sets in the theory of differential games [2] is ensured by estimate (4.17) for sets of initial positions $\{t, \zeta_t(\cdot)\}$ from which the significance of the criterion $\varepsilon^\circ(t, \cdot) \leqslant \varepsilon^\circ(t_0, \cdot) = \varepsilon^\circ(t_0, \zeta_{t_0}(\cdot))$ is guaranteed in the class of program controls. It is not the purpose of the present paper to discuss the corresponding formalization.

5. Solution of the problem. From estimate (4.17) at once follows the existence of the Δ-strategy $U_1(t, \tau_i, \zeta_{\tau_i}(\cdot))$ yielding the result

$$\varepsilon^\circ(\vartheta, \zeta_\vartheta(\cdot)) \leqslant \varepsilon^\circ(t_0, \zeta_{t_0}(\cdot)) + \alpha$$

where the quantity α can be made as small as desired by an appropriate subdivision of the interval $[t_0, \vartheta]$. On each subdivision interval $\tau \in [\tau_i, \tau_{i+1})$ we can take a constant control $u^*(\tau) \equiv u_*$ which should be chosen from conditions (4.14), (4.16). The following assertion is valid.

Theorem 5.1. Suppose that the position $\{t_0, \zeta_{t_0}(\cdot)\}$ is given, that $\varepsilon^\circ(t_0, \zeta_{t_0}(\cdot)) > 0$, and that Conditions 4.1, 4.2 are satisfied. Then for any $\alpha > 0$ there exist a number $\Delta_\alpha > 0$ and a Δ-strategy $U_1(t, \tau_i, \zeta_{\tau_i}(\cdot))$ ensuring, by virtue of inequality (1.8), the condition

$$\varepsilon^\circ(\vartheta, \zeta_\vartheta(\cdot)) = \Phi^\circ(\vartheta, \cdot) \leqslant \varepsilon^\circ(t_0, \zeta_{t_0}(\cdot)) + \alpha$$

for $\Delta < \Delta_\alpha$.

A constructive description of the strategy indicated is given by conditions (4.14), (4.16), (3.4), (2.1). Relation (2.1) was described in detail in [3, 4]. Let the set $\Gamma(t, \cdot)$ consist of the single element $l^\circ(t, \cdot)$ for each point of region $\varepsilon^\circ(t, \cdot) > 0$. Condition 4.2 is ensured by controls $u^{(e)}$ satisfying the maximum condition (see (4.14))

$$l^{\circ\prime}(t, \cdot)X(\vartheta, t)B(t)u^{(e)} = \max_u l^{\circ\prime}(t, \cdot)X(\vartheta, t)B(t)u \qquad (5.1)$$

The convex set $U^{(e)}(t, \cdot) = \{u^{(e)}\}$ of all extremal elements of problem (5.1) is uppersemicontinuous along the directions $\{\Delta t, \Delta\zeta_t(\cdot)\}$ along which $\varepsilon^\circ(t, \cdot)$ is continuous.

From the properties of $X(t, \cdot)$ (see Lemma 2.2) it follows that the set $\{t_i\}$ of jumps of $X(t, \cdot)$ is not more than countable for each realization $y_\vartheta[\cdot]$ depending on $v[t]$, $\xi[t]$, $u[t]$; the instants at which the jumps appear depend only on $v[t]$, $\xi[t]$. In turn, the realization $\varepsilon^\circ[t] = \varepsilon^\circ(t, \zeta_t(\cdot))$ generated by $y[t]$, $u[t]$ can have jumps only on the set $\{t_i\}$. The latter guarantees the existence of a solution of system (1.4)–(1.6) when $U(t, \cdot) = U^{(e)}(t, \cdot)$. The optimality of $U^{(e)}(t, \cdot)$ is ensured by the equality $d\varepsilon^\circ(t, \cdot) / dt \leqslant 0$ following from (5.1) and valid along any of the motions of (1.4)–(1.6) generated by $U^{(e)}(t, \cdot)$ and $y_t[\cdot]$. Thus, the strategy $U^{(e)}(t, \cdot)$ constructed is an optimal x-strategy because the functional $U^\circ(t, X(t, \cdot)) = U^{(e)}(t, \cdot)$ is semicontinuous in $\{t, X\}$ but not in $\{t, \zeta_t(\cdot)\}$.

Theorem 5.2. Suppose that the position $\{t_0, \zeta_{t_0}(\cdot)\}$ is given, that $\varepsilon^\circ(t_0, \zeta_{t_0}(\cdot)) > 0$, and that the sets $\Gamma(t, \cdot)$ contain one element each for each of the positions $\{t, \zeta_t(\cdot)\}$. Then $U^\circ(t, X(t, \cdot))$ is an optimal x-strategy ensuring, by virtue of (1.8), the condition

$$\varepsilon^\circ(\vartheta, \zeta_\vartheta(\cdot)) = \Phi^\circ(\vartheta, \cdot) \leqslant \varepsilon^\circ(t_0, \zeta_{t_0}(\cdot))$$

Finally, we note that for ideally observable systems [11] the set $X(t, \cdot)$ consists of one point $x^*(t, \cdot)$ and $U^\circ = U^\circ(t, x^*(t, \cdot))$.

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GENERALIZED PROBLEM OF THE IMPULSE PURSUIT OF A POINT WITH BOUNDED THRUST

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We consider a game problem [1, 2] similar in formulation to the problems in [3 – 5] and being a direct continuation of the results in [6]. Two material points of unit mass (the first and second players) move in a three-dimensional space under the action of controls F_1, F_2 alone. The control $u = F_1$ is bounded in total momentum, while the control $-v = F_2$ is bounded in absolute value. The game termination set M is an arbitrary fixed point in the space of relative positions and velocities of the players, while the payoff is the time taken to lead a relative trajectory to this point. The first player minimizes this time and the second maximizes it. The solution is in many respects analogous to the solution in [6] wherein the minimax time up to "hard" (with respect to the coordinates) and "soft" (with respect to the coordinates and velocities) contact of the points was determined. In the conclusion we consider the problem of soft contact of two controlled points in a linear position central gravity field. In the course of solving the problem in the title we form a vector-valued function $q(w, p)$ depending upon the game's position w and on a parameter p, and we divide the whole space W of possible positions into the regions W° and W_0. In region W° there exists a function $p_2(w) < 0$, defined as the smallest root of the equation $q(w, p) = 0$.